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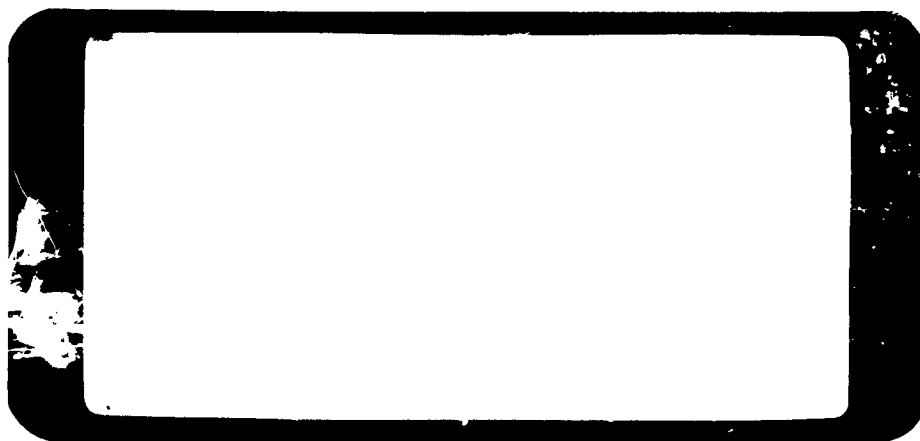
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RESEARCH AND DEVELOPMENT LABORATORIES

HUGHES AIRCRAFT COMPANY

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HUGHES AIRCRAFT COMPANY
Research and Development Laboratories

DOLPH ARRAYS OF MANY ELEMENTS

Technical Memorandum No. 320

By
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August 18, 1953

AF19(604)-262-F-12

This technical memorandum is one of a series of reports which will provide final information available on Project 532-A.

ABSTRACT

G. J. van der Maas' expression for the limiting envelope of the excitation coefficients of a Dolph array as the number of elements is indefinitely increased is derived from the "ideal space factor" function, $\cos \pi \sqrt{u^2 - A^2}$. The envelope is:

$$g_e(p, A) = \frac{\pi A^2}{4} \cdot \frac{2J_1 \left[iA \sqrt{\pi^2 - p^2} \right]}{iA \sqrt{\pi^2 - p^2}}$$

where $\cosh \pi A$ is the side lobe ratio, and p is a variable running from $-\pi$ to π . An envelope for a 40 db side lobe ratio is calculated and compared with the actual Dolph coefficients in a 24 element 40 db array.

I. INTRODUCTION

Ever since C. L. Dolph developed a method for calculating linear arrays with uniform side lobes¹ other workers^{2, 3, 4} in the field have endeavored to perfect this method from the point of view of enhancing the neatness of the expressions and of reducing the labor of the calculations. In particular it has been observed that, for a given side lobe ratio, the excitation coefficient values when plotted with respect to the (normalized) displacement of the corresponding element from the center of the array tend to lie on the same smooth curve (hereafter called the envelope of the excitation coefficients) as M , the total number of elements, is increased. The tendency of the coefficient values to lie on the same envelope is defective only near the ends of the array. The end elements themselves do not appear to be related to the envelope at all and, under some conditions, attain values considerably higher than those of the neighboring elements.

G. J. van der Maas has announced a simple analytic expression for this envelope⁴, which expression was quoted in the abstract. A recent study⁵ of line sources has provided a presumably alternative approach to this same problem by introducing the function $\cos \pi \sqrt{u^2 - A^2}$ which has a main lobe of adjustable height and an infinite number of equal side lobes. This function is known as the "ideal space factor" and, by its use, the envelope of the Dolph coefficients can be very easily derived. The purpose of this memorandum is to make this derivation.

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1. C. L. Dolph, Proc. Inst. Radio Engrs. 34, 335-348 (June 1946).
 2. D. Barbieri, Proc. Inst. Radio Engrs. 40, 78-82 (January 1952).
 3. R. J. Stegen, "Excitation Coefficients and Beamwidths of Tschebyscheff Arrays." Accepted for publication, Proc. Inst. of Radio Engrs.
 4. G. J. van der Maas, "A Simplified Calculation for Dolph-Tschebyscheff Arrays," Symposium on Microwave Optics, McGill University, Montreal, Canada (June 1953).
 5. T. T. Taylor, "Design of Line Sources for Narrow Beamwidth and Low Side Lobes," Hughes Aircraft Company Technical Memorandum No. 316.

II. THE DOLPH SPACE FACTOR

Let the following definitions be made:

M = number of elements in array

d = interelement spacing

$\ell = Rd$ = effective length⁶; $R \doteq M$ and $\frac{R}{M} \rightarrow 1$ as $M \rightarrow \infty$

$k = 2\pi/\lambda$ = free space wave number

θ = observer's angle measured from end of array

$\nu = \cos \theta$

$\psi = kd \nu$

The space factor of a Dolph array is then given by:

$$S_M(\psi) = T_{M-1}(B \cos \frac{\psi}{2}) \quad (1)$$

or by:

$$S_M(\nu) = T_{M-1}(B \cos \frac{\pi \ell \nu}{R\lambda}) \quad (2)$$

where:

$T_{M-1}(w)$ = Tschebyscheff polynomial of order $M-1$

$$w = B \cos \frac{\psi}{2}$$

$$B = \cosh \left[\frac{1}{M-1} \operatorname{arc} \cosh \eta \right] = "Z_0" \text{ of Dolph's original paper}$$

η = side lobe voltage ratio

$$\frac{\psi}{2} = "u" \text{ of Dolph's original paper.}$$

(u is not used again in this sense here)

Equation (1) is most easily explained by Figure 1, which depicts $T_4(w)$ and a simple geometrical construction⁷ showing how $S_5(\psi)$ can be obtained. Now for convenience let:

$$u = \operatorname{Re} z = \frac{\ell \nu}{\lambda} = \frac{R \psi}{2\pi} \quad (3)$$

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6. The effective length of a linear array will be defined as the length of the smooth continuous approximating distribution which produces a pattern of the same beamwidth as that of the given array. For the present it is sufficient to observe that R and M are approximately equal.
 7. This is not to be confused with the very similar geometrical construction relating $S(\theta)$ to $S(\psi)$.

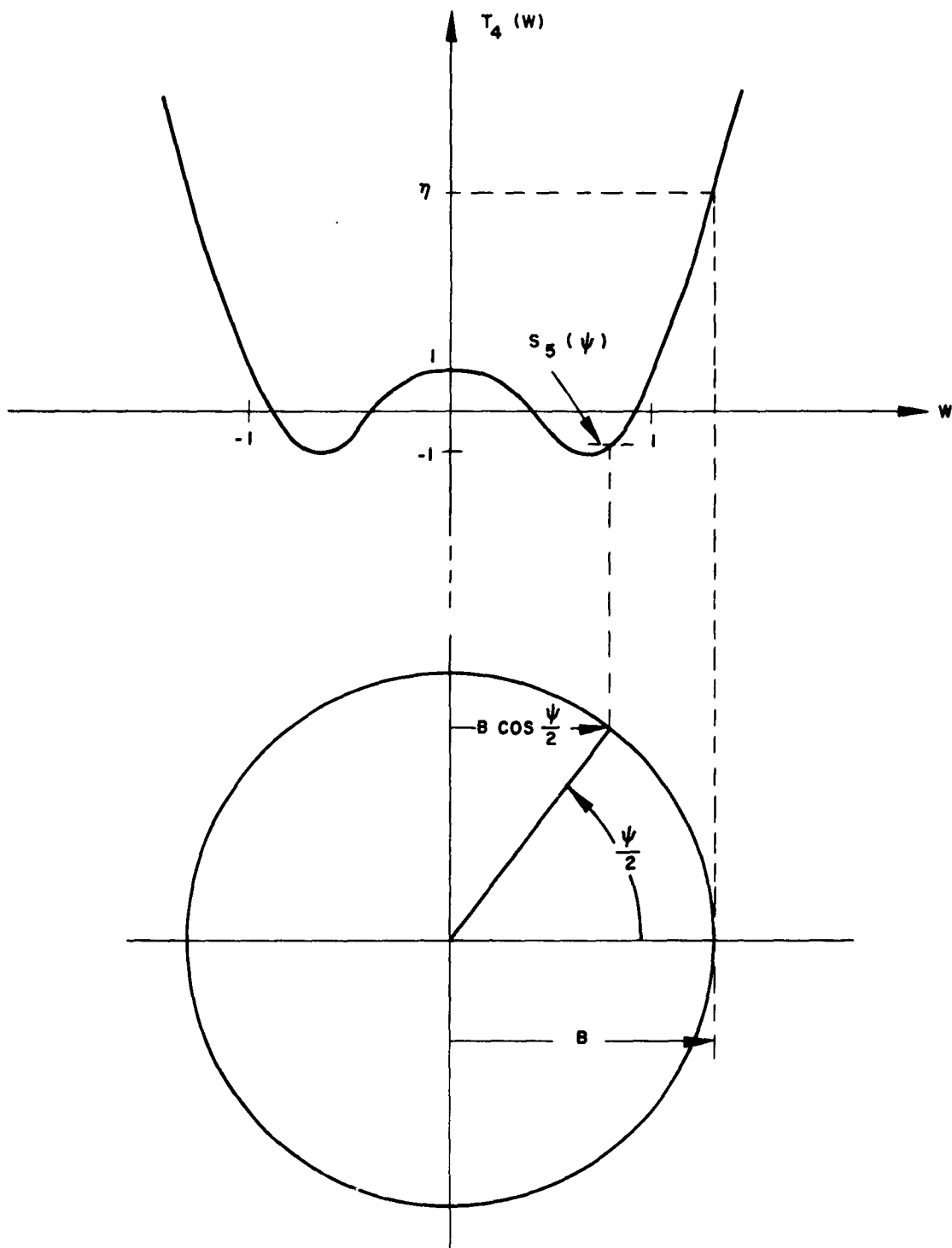


FIGURE 1

These are the same u and z as those of Reference 5; a unit increment in u corresponds to a standard beamwidth, λ/\mathcal{L} . In this memorandum u will be used in preference to z since the real-variable behaviour is emphasized here. The space factor becomes:

$$S_M(u) = T_{M-1} \left(B \cos \frac{\pi u}{R} \right). \quad (4)$$

The zeros, u_n , of $S_M(u)$ occur when the argument fulfills the following condition:

$$B \cos \frac{\pi u_n}{R} = \cos \left[\frac{-\pi}{2(M-1)} + \frac{n\pi}{M-1} \right] \quad (5)$$

$$n = 1, 2, 3 \dots M-1$$

It is now necessary to make some remarks about the periodicity of $S_M(u)$. Evidently $S_M(\psi)$ has a period of 2π in ψ if M is odd and a period of 4π if M is even. In either event, $|S_M(\psi)|$ has a period of 2π as demanded by array theory⁸. Similarly $|S_M(u)|$ has a period of R in u . Hence, if M , and therefore R , are increased indefinitely the period of the space factor in u space increases indefinitely and the secondary maxima become more and more remote. As M tends to infinity, the formula for the finite zeros becomes:

$$\left[1 + \frac{1}{2} \left(\frac{\text{arc cosh } \eta}{M-1} \right)^2 \right] \left[1 - \frac{1}{2} \left(\frac{\pi u_n}{R} \right)^2 \right] = 1 - \frac{1}{2} \left(\frac{-\pi}{2(M-1)} + \frac{n\pi}{M-1} \right)^2 \quad (6)$$

Neglecting higher order terms in $1/M$:

$$(\text{arc cosh } \eta)^2 - (\pi u_n)^2 = - \left(n - \frac{1}{2} \right)^2 \pi^2 \quad (7)$$

Whence:

$$u_n = \pm \sqrt{\left(\frac{\text{arc cosh } \eta}{\pi} \right)^2 + \left(n - \frac{1}{2} \right)^2} \quad (8)$$

As in Reference 5, the following abbreviation will be made:

$$\pi A = \text{arc cosh } \eta \quad (9)$$

8. T. T. Taylor and J. R. Whinnery, Journal of Applied Physics 22, 19-29 (January 1951).

Whereupon the zeros are given by

$$\left. \begin{aligned} u_n &= \pm \sqrt{A^2 + (n - \frac{1}{2})^2} \\ 1 \leq n < \infty \end{aligned} \right\} \quad (10)$$

The profile on the real axis of the entire function with these zeros is shown in Reference 5 to be:

$$\boxed{F_0(u, A) = \cos \pi \sqrt{u^2 - A^2}} \quad (11)$$

Hence the function $F_0(u, A)$ can be regarded as the limiting form of the space factor of a Dolph array as the number of elements is indefinitely increased. In the subsequent section, this function will be utilized in obtaining the envelope of the excitation coefficients.

III. THE ENVELOPE

It has just been shown that the limiting form (as $M \rightarrow \infty$) of the space factor of a Dolph array is a very simple function of u . The inverse Fourier transform of this function should provide some information relevant to the envelope of the excitation coefficients. Unfortunately this inverse transform is unbounded since the integral from $-\infty$ to ∞ of $[F_0(u, A)]^2$ is unbounded. A simple trick will overcome this difficulty, however. Write:

$$\left. \begin{aligned} F_0(u, A) &= \left[\cos \pi \sqrt{u^2 - A^2} - \cos \pi u \right] + \cos \pi u \\ &= F_e(u, A) + \cos \pi u \end{aligned} \right\} \quad (12)$$

The inverse transform of $F_e(u, A)$ is bounded and is formally given by:

$$g_e(p, A) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_e(u, A) e^{-ipu} du \quad (13)$$

This integral is known⁹ to give the simple function:

9. Campbell and Foster, Fourier Integrals, D. Van Nostrand Co., Inc., New York, 1948. See transform No. 871.2.

$$g_e(p, A) = \begin{cases} \frac{\pi A^2}{4} \cdot \frac{2J_1 \left[iA \sqrt{\pi^2 - p^2} \right]}{iA \sqrt{\pi^2 - p^2}} & p^2 \leq \pi^2 \\ 0 & p^2 > \pi^2 \end{cases} \quad (14)$$

Here J_1 is a Bessel function of the first kind. Its values for imaginary argument are conveniently tabulated in Jahnke and Emde¹⁰, and elsewhere^{11,12}. The variable p is $2\pi x/\ell$ where x measures physical distance from the center of the distribution, as in Reference 5.

To find the total inverse transform, $g_o(p, A)$, it is necessary to add to the above the inverse transform of $\cos \pi u$:

$$\left. \begin{aligned} g_s(p) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos \pi u e^{-ipu} du \\ &= \lim_{r \rightarrow \infty} \frac{1}{2\pi} \int_{-r}^r \cos \pi u e^{-ipu} du \\ &= \lim_{r \rightarrow \infty} \frac{r}{2\pi} \left\{ \frac{\sin r(p-\pi)}{r(p-\pi)} + \frac{\sin r(p+\pi)}{r(p+\pi)} \right\} \end{aligned} \right\} \quad (15)$$

$$g_s(p) = \frac{1}{2} [\delta(p-\pi) + \delta(p+\pi)] \quad (16)$$

Thus it is seen that the inverse transform of $\cos \pi u$ is a pair of impulse functions located at $p = -\pi$ and at $p = \pi$. The total inverse transform is then:

$$g_o(p, A) = \begin{cases} \frac{\pi A^2}{4} \cdot \frac{2J_1 \left[iA \sqrt{\pi^2 - p^2} \right]}{iA \sqrt{\pi^2 - p^2}} + \frac{1}{2} [\delta(p-\pi) + \delta(p+\pi)] & p^2 \leq \pi^2 \\ 0 & p^2 > \pi^2 \end{cases} \quad (17)$$

10. Jahnke and Emde, Tables of Functions, Dover Publications, New York, 1943. See page 227 et seq.

11. British Association Mathematic Tables, Vols. VI and X, The University Press, Cambridge, 1952.

12. National Bureau of Standards, Tables of $J_0(z)$ and $J_1(z)$ for Complex Argument, Columbia University Press, New York, 1943.

Evidently $g_o = g_e$ in the open interval $-\pi < p < \pi$. Hence $g_e(p, A)$ is the limiting case of the envelope of the internal excitation coefficients as M is increased indefinitely. The end excitation coefficients which -- as has been observed before -- do not conform to the envelope, become the impulse functions at the two ends of the distribution.

IV. AN EXAMPLE

It is rather natural to assume that the end elements of a Dolph array of a large number of elements should coincide with the end points of the envelope function if the two are superposed. (This must happen as $M \rightarrow \infty$.) Proceeding on this basis the envelope function $g_e(p, A)$ for a 40 db side lobe ratio has been calculated and is plotted in Figure 2; calculated values of the excitation coefficients for a 24 element Dolph array are displayed on the same graph for comparison. The excitation coefficients were normalized so as to be equal to the envelope at the center of the array and the close agreement between the envelope and the calculated excitation coefficients in other parts of the graph should be noted.

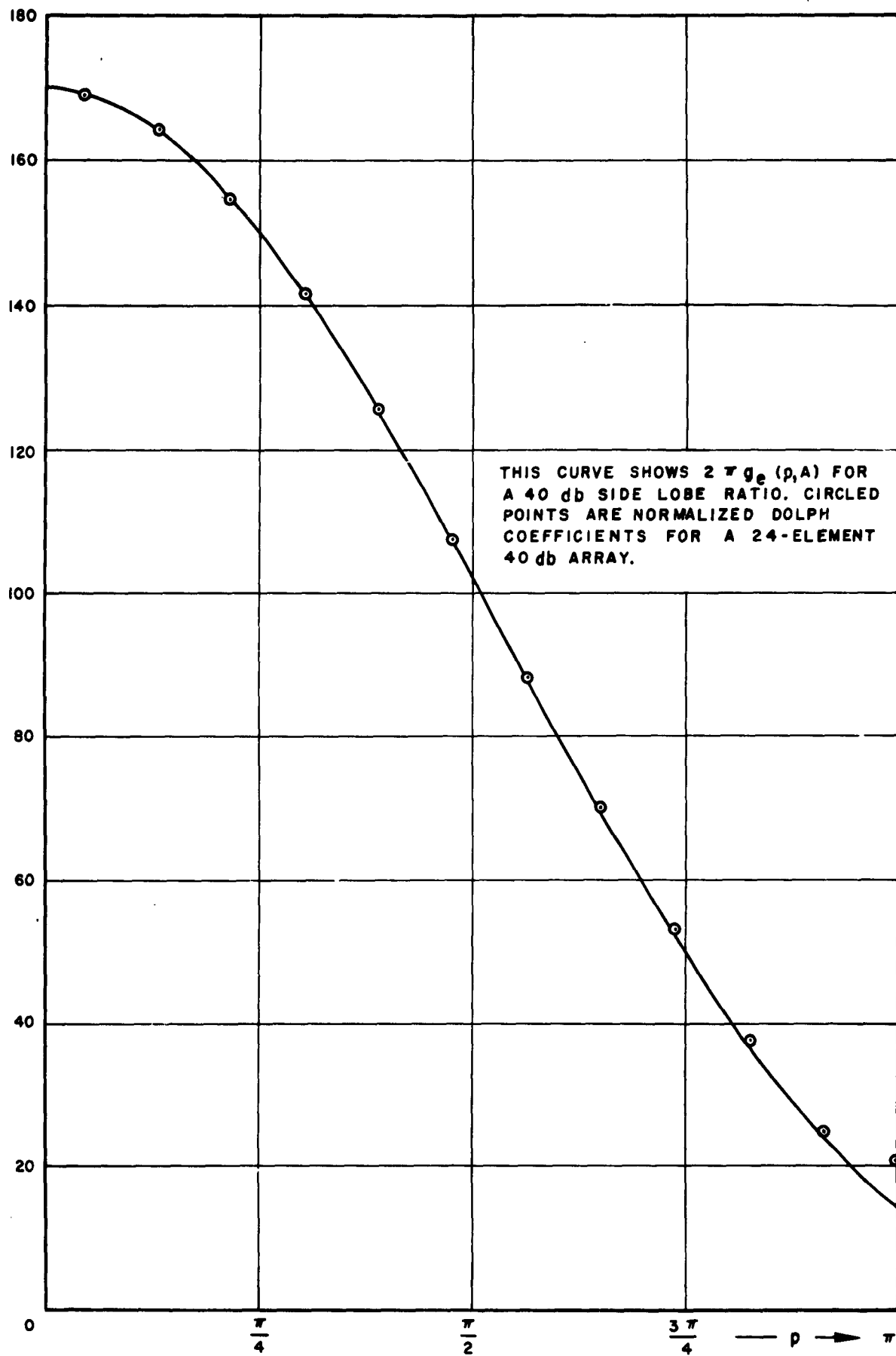


FIGURE 2

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